Solutions to Theory Problem 3

Physics of Live Systems

(Rui Travasso, Lucília Brito)

July 24, 2018
Physics of Live Systems (10 points)

Part A. The physics of blood flow (4.5 points)

A.1

Since the vessel network is symmetrical, the flow in a vessel of level \( i + 1 \) is half the flow in a vessel of level \( i \).

In this way, we can sum the pressure differences in all levels:

\[
\Delta P = \sum_{i=0}^{N-1} Q_i R_i = Q_0 \sum_{i=0}^{N-1} \frac{R_i}{2^i}.
\]

Introducing the radii dependences yields

\[
\Delta P = Q_0 \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = Q_0 \frac{8\ell_0 \eta}{\pi r_0^4} \sum_{i=0}^{N-1} \frac{2^{4i/3}}{2^{2i/3}} = Q_0 N \frac{8\ell_0 \eta}{\pi r_0^4}.
\]

Therefore

\[
Q_0 = \frac{\Delta P \pi r_0^4}{8N \ell_0 \eta}.
\]

Hence, the flow rate for a vessel network in level \( i \) is

\[
Q_i = \frac{\Delta P \pi r_0^4}{2^{4i+3} N \ell_0 \eta}.
\]

1.3pt

A.2

Replace values in the formula and change units appropriately

\[
Q_0 = \frac{\Delta P \pi r_0^4}{8N \ell_0 \eta} = \frac{(55 - 30) \times 1.013 \times 10^5 \times 3.1415 \times (6.0 \times 10^{-5})^4}{760 \times 48 \times 2.0 \times 10^{-3} \times 3.5 \times 10^{-3}} = 4.0 \times 10^{-10} \text{ m}^3/\text{s}
\]

to obtain the final value in the requested unites:

\[
Q_0 \approx 1.5 \text{ m}^3/\text{h}.
\]

0.5pt
A.3

The current is given by

\[ I = \frac{P_{in}e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}}. \]

The pressure difference in the capacitor is

\[ P_{out}e^{i(\omega t + \phi)} = \frac{P_{in}e^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \cdot \frac{1}{i\omega C} = \frac{P_{in}e^{i\omega t}}{i\omega CR - \omega^2 LC + 1}. \]

The amplitude is

\[ P_{out} = \frac{P_{in}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}}. \]

To be smaller than \( P_{in} \), for \( \omega \to 0 \):

\[ (1 - \omega^2 LC)^2 + \omega^2 C^2 R^2 > 1 \iff -2CL + C^2 R^2 > 0. \]

Replacing the expressions for \( L, C, \) and \( R \) we get: \( \frac{64\eta^2 \ell^2}{3Er^4p} > 1. \)

**Condition:**

\[ \frac{64\eta^2 \ell^2}{3Er^4p} > 1. \]

Alternative way to obtain \( P_{out} \):

The amplitude of the current in the equivalent circuit is \( I_0 = \frac{P_{in}}{Z} \), where

\[ Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \]

is the modulus of the impedance. Hence, the voltage amplitude in the capacitor is

\[ P_{out} = \frac{1}{\omega C} \times I_0 = \frac{P_{in}}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}}. \]

A.4

The previous condition can also be expressed as

\[ h < \frac{64\eta^2 \ell^2}{3Er^4p}. \]

For the network referred to in A.2

\[ h < \frac{64\eta^2 \ell^2}{3 \times 2^{2/3} Er^4p} = \frac{64 \times (3.5 \times 10^{-3})^2 \times (2.0 \times 10^{-3})^2}{3 \times 0.06 \times 10^6 \times (6.0 \times 10^{-5})^3 \times 1.05 \times 10^4} \times 2^{2/3} = 7.7 \times 10^{-5} \times 2^{2/3}. \]
For $i = 0$, in the worse case scenario,

$$h_{\text{max}} = 7.7 \times 10^{-5} \times 2^0 = 7.7 \times 10^{-5} \text{ m}$$

This value is certainly observed in these vessels since their radius range from 18 $\mu$m to 60 $\mu$m. A wall width smaller than 80 $\mu$m is certainly reasonable.

A.4 Maximum $h = 8 \times 10^{-5}$ m

Part B. Tumor growth (5.5 points)

B.1

The expressions for the masses of tumour and normal tissue are written as:

$$\begin{align*}
M_T &= V_T \rho_T = V_T \rho_0 (1 + \frac{\rho}{\rho_T}) \\
M_N &= V \rho_0 = (V - V_T) \rho_0 (1 + \frac{\rho}{\rho_N})
\end{align*}$$

The pressure, $p$, can be expressed as

$$p = \frac{M_T K_T}{V_T \rho_0} - K_T$$

and, then, used in the equation for $M_N$:

$$M_N = (V - V_T) \frac{M_N}{V} \left[ \left( 1 - \frac{K_T}{K_N} \right) + \frac{M_T V K_T}{V_T M_N K_N} \right]$$

Simplifying and rearranging the terms, the equation for $v$ becomes

$$(1 - \kappa) v^2 - (1 + \mu) v + \mu = 0,$$

for which the solution is (the other solution of the quadratic equation is not physically relevant since does not lead to $v = 0$ for $\mu = 0$)

B.1

$$v = \frac{1 + \mu - \sqrt{(1 + \mu)^2 - 4\mu (1 - \kappa)}}{2(1 - \kappa)}.$$

B.2

For $r < R_T$, the conservation of energy implies that

$$4\pi r^2 (-k) \frac{dT}{dr} = \rho \frac{A}{3} \pi r^3.$$
Therefore, the temperature difference to 37 °C = 310.15 K, \( \Delta T(r) \), is given by

\[
\Delta T(r) = -\frac{\rho r^2}{6k} + C,
\]

where \( C \) is a constant.

For \( r > R_T \), the conservation of energy implies that

\[
4\pi r^2(-k) \frac{dT}{dr} = \frac{\rho}{3\pi R_T^3}.
\]

Therefore, the temperature difference to 37 °C is

\[
\Delta T(r) = \frac{\rho R_T^3}{3kr}.
\]

In this case there is no constant, since very far away the increase in temperature is zero.

Matching the two solutions at \( r = R_T \) gives

\[
C = \frac{\rho R_T^2}{2k}.
\]

Therefore the temperature at the centre of the tumour, in SI units, is

\[
B.2 \quad \text{Temperature: } 310.15 + \frac{\rho R_T^2}{2k}.
\]

\[
B.3 \quad \rho_{min} = 4.3 \text{ kW/m}^3.
\]

\[
B.4 \quad \text{The pressure can be related with the volume. We know that}
\]

\[
\frac{M_N}{V_N} = \frac{\rho_0 V}{V - V_T} = \frac{\rho_0}{1 - v} = \rho_0 \left( 1 + \frac{p}{K_N} \right).
\]
And so \( p = \frac{K_n v}{1 - v} \).

When the thinner vessels are narrower, the flow rate in the main vessel is altered:

\[
\Delta P = (Q_0 + \delta Q_0) \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2\pi r_i^4} = (Q_0 + \delta Q_0) \frac{8\ell_0 \eta}{\pi r_0^4} \left( \sum_{i=0}^{N-2} \frac{2^{4i/3}}{2^i 2^{1/3}} + \frac{2^{4(N-1)/3}}{2^{N-1}2(N-1)/3} \left( 1 - \frac{\delta r}{r_0^{2(N-1)/3}} \right)^4 \right)
\]

\[
\implies \Delta P \approx (Q_0 + \delta Q_0) \frac{\Delta P}{NQ_0} \left( N - 1 + 1 + \frac{4\delta r}{r_{N-1}} \right)
\]

Noting that \( \frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0} \), we obtain

\[
1 + \frac{\delta Q_{N-1}}{Q_{N-1}} = 1 + \frac{4\delta r}{N r_{N-1}} \approx 1 - \frac{4\delta r}{N r_{N-1}}.
\]

And so:

\[
\frac{\delta Q_{N-1}}{Q_{N-1}} \approx - \frac{4\delta r}{N r_{N-1}}.
\]

Putting all together

\[
\delta Q_{N-1} \approx - \frac{2}{N} \frac{K_n v - (1 - v)P_{cap}}{(1 - v)(p_c - P_{cap})} \frac{4\delta r}{r_{N-1}}.
\]