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Solutions to Theory Problem 1

LIGO-GW150914

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v6.0

GW150914 (10 points)

Part A. Newtonian (conservative) orbits (3.0 points)

A.1 Apply Newton's law to mass M_1:

$$M_1 \frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = G\frac{M_1 M_2}{|\vec{r}_2-\vec{r}_1|^2}\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|} . \tag{1}$$

Use, from eq. (1) of the question sheet

$$\vec{r}_2 = -\frac{M_1}{M_2}\vec{r}_1 , \tag{2}$$

in eq. (1) above, to obtain

$$\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = -\frac{GM_2^3}{(M_1+M_2)^2 r_1^2}\frac{\vec{r}_1}{r_1} . \tag{3}$$

A.1	1.0pt
$$n = 3, \qquad \alpha = \frac{GM_2^3}{(M_1+M_2)^2} .$$	

A.2 The total energy of the system is the sum of the two kinetic energies plus the gravitational potential energy. For circular motions, the linear velocity of each of the masses reads

$$|\vec{v}_1| = r_1\Omega , \qquad |\vec{v}_2| = r_2\Omega , \tag{4}$$

Thus, the total energy is

$$E = \frac{1}{2}(M_1 r_1^2 + M_2 r_2^2)\Omega^2 - \frac{GM_1M_2}{L} , \tag{5}$$

Now,

$$(M_1 r_1 - M_2 r_2)^2 = 0 \qquad \Rightarrow \qquad M_1 r_1^2 + M_2 r_2^2 = \mu L^2 . \tag{6}$$

Thus,

$$E = \frac{1}{2}\mu L^2\Omega^2 - G\frac{M\mu}{L} . \tag{7}$$

A.2	1.0pt
$$A(\mu,\Omega,L) = \frac{1}{2}\mu L^2\Omega^2 .$$	

A.3 Energy (3) of the question sheet can be interpreted as describing a system of a mass μ in a circular orbit with angular velocity Ω, radius L, around a mass M (at rest). Equating the gravitational acceleration to the centripetal acceleration:

$$G\frac{M}{L^2} = \Omega^2 L . \tag{8}$$

This is indeed Kepler's third law (for circular orbits). Then, from (7),

$$E = -\frac{1}{2}G\frac{M\mu}{L} . \tag{9}$$

A.3	1.0pt
$$\beta = -\frac{1}{2} .$$	

Part B - Introducing relativistic dissipation (7.0 points)

B.1 Some simple trigonometry for the x, y motion of the masses (in an appropriate Cartesian system) yields:

$$(x_1, y_1) = r_1(\cos(\Omega t), \sin(\Omega t)) , \qquad (x_2, y_2) = -r_2(\cos(\Omega t), \sin(\Omega t)) . \tag{10}$$

Then,

$$Q_{ij} = \frac{M_1 r_1^2 + M_2 r_2^2}{2} \begin{pmatrix} \frac{4}{3}\cos^2(\Omega t) - \frac{2}{3}\sin^2(\Omega t) & 2\sin(\Omega t)\cos(\Omega t) & 0 \\ 2\sin(\Omega t)\cos(\Omega t) & \frac{4}{3}\sin^2(\Omega t) - \frac{2}{3}\cos^2(\Omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} , \tag{11}$$

or, using some simple trigonometry and (6),

$$Q_{ij} = \frac{\mu L^2}{2} \begin{pmatrix} \frac{1}{3} + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & \frac{1}{3} - \cos 2\Omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} . \tag{12}$$

B.1	1.0pt
$k = 2\Omega \,, \quad a_1 = a_2 = \dfrac{1}{3}\,, a_3 = -\dfrac{2}{3}, \quad b_1 = 1, b_2 = -1, b_3 = 0\,, c_{12} = c_{21} = 1, c_{ij} \overset{\text{otherwise}}{=} 0\,.$	

B.2 First take the derivatives:

$$\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} = 4\Omega^3 \mu L^2 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{13}$$

Then perform the sum:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}(4\Omega^3\mu L^2)^2[2\sin^2(2\Omega t) + 2\cos^2(2\Omega t)] = \frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{14}$$

B.2	1.0pt
$\xi = \dfrac{32}{5}\,.$	

B.3 Now we assume a sequency of Keplerian orbits, with decreasing energy, which is being taken from the system by the GWs.

First, from (9), differentiating with respect to time,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} , \tag{15}$$

Since this loss of energy is due to GWs, we equate it with (minus) the luminosity of GWs, given by (14)

$$\frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{16}$$

We can eliminate the L and $\mathrm{d}L/\mathrm{d}t$ dependence in this equation in terms of Ω and $\mathrm{d}\Omega/\mathrm{d}t$, by using Kepler's third law (8), which relates:

$$L^3 = G\frac{M}{\Omega^2} , \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{2}{3}\frac{L}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}t} . \tag{17}$$

Substituting in (16), we obtain:

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} G^5 \mu^3 M^2 \equiv \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} \left(GM_{\mathsf{c}}\right)^5 . \tag{18}$$

B.3	1.0pt
$M_{\mathsf{c}} = (\mu^3 M^2)^{1/5}$.	

B.4 Angular and cycle frequencies are related as $\Omega = 2\pi f$. From the information provided above: *GWs have a frequency which is twice as large as the orbital frequency*, we have

$$\frac{\Omega}{2\pi} = \frac{f_{\mathsf{GW}}}{2} . \tag{19}$$

Formula (10) of the question sheet has the form

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \chi \Omega^{11/3} , \qquad \chi \equiv \frac{96}{5} \frac{(GM_{\mathsf{c}})^{5/3}}{c^5} . \tag{20}$$

Thus, from (11) of the question sheet

$$\Omega(t)^{-8/3} = \frac{8}{3}\chi(t_0 - t) , \tag{21}$$

or, using (20) and the definition of χ gives

$$f_{\mathsf{GW}}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{GM_{\mathsf{c}}}{c^3}\right)^{5/3}(t_0 - t) . \tag{22}$$

B.4	2.0pt
$p = 1$.	

B.5 From the figure, we consider the two Δt's as half periods. Thus, the (cycle) GW frequency is $f_{\mathsf{GW}} = 1/(2\Delta t)$. Then, the four given points allow us to compute the frequency at the mean time of the two intervals as

	t_{AB}	t_{CD}
t (s)	0.0045	0.037
f_{GW} (Hz)	$(2 \times 0.009)^{-1}$	$(2 \times 0.006)^{-1}$

Now, using (22) we have two pairs of (f_{GW},t) values for two unknowns (t_0,M_{c}). Expressing (22) for both t_{AB} and t_{CD} and dividing the two equations we obtain:

$$t_0 = \frac{A t_{\mathsf{CD}} - t_{\mathsf{AB}}}{A - 1} , \qquad A \equiv \left(\frac{f_{\mathsf{GW}}(t_{\mathsf{AB}})}{f_{\mathsf{GW}}(t_{\mathsf{CD}})}\right)^{-8/3} . \tag{23}$$

Replacing by the numerical values, $A \simeq 2.95$ and $t_0 \simeq 0.054$ s. Now we can use (22) for either of the two values t_{AB} or t_{CD} and determine M_{c}. One obtains for the chirp mass

$$M_{\mathsf{c}} \simeq 6 \times 10^{31}\ \text{kg} \simeq 30 \times M_{\odot} . \tag{24}$$

Thus, the total mass M is

$$M = 4^{3/5} M_{\mathsf{c}} \simeq 69 \times M_{\odot} . \tag{25}$$

This result is actually remarkably close to the best estimates using the full theory of General Relativity! [Even though the actual objects do not have precisely equal masses and the theory we have just used is not valid very close to the collision.]

B.5	1.0pt

$$M_{\mathsf{c}} \simeq 30 \times M_\odot \,, \qquad M \simeq 69 \times M_\odot \,.$$

B.6 From (8), Kepler's law states that $L = (GM/\Omega^2)^{1/3}$. The second pair of points highlighted in the plot correspond to the cycle prior to merger. Thus, we use (19) to obtain the orbital angular velocity at $t_{\overline{\mathsf{CD}}}$:

$$\Omega_{t_{\overline{\mathsf{CD}}}} \sim 2.6 \times 10^2 \text{ rad/s} \,. \tag{26}$$

Then, using the total mass (25) we find

$$L \sim 5 \times 10^2 \text{ km} \,. \tag{27}$$

Thus, these objects have a maximum radius of $R_{\mathsf{max}} \sim 250$ km. Hence they have over 30 times more mass and,

$$\frac{R_\odot}{R_{\mathsf{max}}} \sim 3 \times 10^3 \tag{28}$$

they are 3000 times smaller than the Sun and!

Their linear velocity is

$$v_{\mathsf{col}} = \frac{L}{2}\Omega \simeq 7 \times 10^4 \text{ km/s} \,. \tag{29}$$

They are moving at over 20% of the velocity of light!

B.6	1.0pt

$$L_{\mathsf{collision}} \sim 5 \times 10^2 \text{ km} \,, \qquad \frac{R_\odot}{R_{\mathsf{max}}} \sim 3 \times 10^3 \,, \qquad \frac{v_{\mathsf{col}}}{c} \sim 0.2 \,.$$