Solutions to Experimental Problem 2

Viscoelasticity of a polymer thread

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Problem 2: Viscoelasticity of a polymer thread (10 points)

Part A. Stress-relaxation measurements (1.9 points)

A.1
Measurement: \( \ell_0 = 42.7 + 2 \times 0.5 = 43.7 \) cm,

\[
\ell_0 = (43.7 \pm 0.2) \text{ cm}.
\]

A.2

\[
P_0 = (81.11 \pm 0.03) \text{ gf}.
\]

A.3
The table contains the readings on the scale \( P \) (Question A.3) and the force on the thread, \( F(t) \), at constant strain (Question D.1). The values of \( \frac{dF}{dt} \) (Question D.6) were computed numerically using equal time intervals. The function \( y(t) \) is given by \( y(t) = F(t) - F_0 - F_1 e^{-t/\tau_1} \) (Question D.10).

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<th>( F / \text{gf} )</th>
<th>( \frac{dF}{dt} / \text{gf s}^{-1} )</th>
<th>( y(t) / \text{gf} )</th>
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</table>
A.4

Measurement: \( \ell = 50.0 + 2 \times 0.5 = 51.0 \text{ cm} \),

\[
\ell = (51.0 \pm 0.2) \text{ cm}.
\]

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Part B. Measurement of the stretched thread diameter (1.5 points)

B.1

Two mirrors are used to maximize the distance \( D \) and consequently the distance between diffraction minima.

B.2

The total distance \( D \) is the sum

\[
D = D_1 + D_2 + D_3 = (26.0 + 36.0 + 102.3) \text{ cm} = 164.3 \text{ cm} = 1.643 \text{ m}.
\]

The estimated uncertainties are

\[
\sigma_{D_1} = \sigma_{D_2} = \sigma_{D_3} \approx 0.5 \text{ cm} \quad \Rightarrow \quad \sigma_D = \sqrt{3 \times \sigma_{D_1}^2} = 0.5 \times \sqrt{3} = 0.87 \text{ cm}.
\]

\[
D = (1.643 \pm 0.009) \text{ m}.
\]
B.3

The distance between minima, \(x\), is quite small. To reduce the error, the total distance \(N\times\), with \(N = 22\), was measured:

\[22\times = 49\text{ mm} \Rightarrow \bar{x} = 2.277\text{ mm}.

The corresponding uncertainty is

\[\sigma_{\bar{x}} = \frac{\sigma_{22\times}}{22} = \frac{0.25\text{ mm}}{22} = 0.011\text{ mm}.

\[\bar{x} = (2.277 \pm 0.011)\text{ mm}.

B.4

Using previous results, we get

\[d = \frac{\lambda}{\sin \theta} \approx \frac{\lambda D}{\bar{x}} = \frac{650 \times 10^{-9}\text{ m} \times 1.643\text{ m}}{2.277 \times 10^{-3}\text{ m}} = 4.795 \times 10^{-4}\text{ m} = 0.480\text{ mm}.

For the uncertainties, we have

\[\frac{\sigma_d}{d} = \frac{\sigma_\lambda}{\lambda} + \frac{\sigma_D}{D} + \frac{\sigma_{\bar{x}}}{\bar{x}} = \frac{10}{650} + \frac{0.0087}{1.643} + \frac{0.011}{2.227} = 0.02517 \Rightarrow \sigma_d = 0.02517 \times 0.480\text{ mm} = 0.012\text{ mm}.

\[d = (0.480 \pm 0.012)\text{ mm}.

Part C. Change to a new thread (0.3 points)

C.1

Measurement: \(\ell_0' = 31.6 + 2 \times 0.5 = 32.6\text{ cm}.

\[\ell_0' = (32.6 \pm 0.2)\text{ cm}.

Part D. Data Analysis (5.7 points)

D.1

The force on the thread was calculated as \( F(t) = (P_0 - P(t)) \), in gram-force units.

D.1 See column \( F(t) \) in the table in A.3.

D.2

Left: \( F(t) \) sampled at unequal time intervals. Right: \( F(t) \) sampled at equal time intervals for \( t > 1000 \) s.

D.3

The dimensionless quantity \( \epsilon \) is given by

\[
\epsilon = \frac{\ell - \ell_0}{\ell_0} = \frac{51.0 - 43.7}{43.7} = 0.167 .
\]

The uncertainty in \( \epsilon \), \( \sigma_\epsilon \), is calculated propagating the uncertainties in the measured length, \( \sigma_\ell \) and \( \sigma_{\ell_0} \):

\[
\frac{\sigma_\epsilon}{\epsilon} = \frac{\sigma_{\ell - \ell_0}}{\ell - \ell_0} + \frac{\sigma_{\ell_0}}{\ell_0} = \sqrt{\frac{\sigma_\ell^2 + \sigma_{\ell_0}^2}{\ell - \ell_0} + \frac{\sigma_{\ell_0}^2}{\ell_0}} = 0.2 \times \frac{\sqrt{2}}{7.3} + \frac{0.2}{43.7} = 0.0433
\]

Therefore, \( \sigma_\epsilon = 0.0433 \times 0.167 = 0.0072 \).

D.3

\[
\epsilon = 0.167 \pm 0.007 .
\]
D.4

One has

\[ \frac{\sigma}{\epsilon} = \frac{F}{S}. \]

In this case, \( S = \pi(d/2)^2 = 1.809 \times 10^{-7} \text{ m}^2 \) and \( \epsilon = 0.167 \). We also have 1 gf = \( g \times 10^{-3} \) N with \( g = 9.8 \text{ m s}^{-2} \). Therefore, if \( F \) is in gram-force units we have

\[ \frac{\sigma}{\epsilon} = \frac{9.8 \times 10^{-3} \text{ gf}^{-1} \text{ N}}{0.167 \times 1.809 \times 10^{-7} \text{ m}^2} F = (324293 \text{ gf}^{-1} \text{ N m}^{-2}) F, \]

where \( F \) is in gf, and \( \sigma \) is in N m\(^{-2}\). Comparing with \( \frac{\sigma}{\epsilon} = \beta F \) we get

\[ \beta = 324293 \text{ gf}^{-1} \text{ N m}^{-2}. \]

Note that, if we write

\[ F(t) = F_0 + F_1 e^{-t/\tau_1} + F_2 e^{-t/\tau_2} + F_3 e^{-t/\tau_3} + \ldots \quad (1) \]

and compare with equation

\[ \frac{\sigma}{\epsilon} = \beta F(t) = E_0 + E_1 e^{-t/\tau_1} + E_2 e^{-t/\tau_2} + E_3 e^{-t/\tau_3} + \ldots \quad (2) \]

we conclude that \( E_0 = \beta F_0, E_1 = \beta F_1, E_2 = \beta F_2, \) etc.

\[ \beta = 3.24 \times 10^5 \text{ gf}^{-1} \text{ N m}^{-2}. \]

D.5

For a purely elastic process, \( \sigma = \epsilon E_0 \) and

\[ F = \alpha \sigma = \alpha \epsilon E_0. \]

Thus, a graph of a constant function is expected.
D.6

The data for $\frac{dF}{dt}$ inserted in table introduced in A.3, was computed numerically for equal time intervals. However, the graphical method is also exemplified. In the present graph, tangent lines to $F(t)$ are drawn at four different time instants (1200, 1500, 1800 and 2100 s). The slopes of those lines are a measure of $\frac{dF}{dt}$ at those instants.

D.7

For a single viscoelastic process,

$$F = \frac{1}{\beta} (E_0 + E_1 e^{-t/\tau_1}) = F_0 + F_1 e^{-t/\tau_1}.$$ 

Therefore,

$$\frac{dF}{dt} = -F_1 \frac{1}{\tau_1} e^{-t/\tau_1}, \quad \text{where} \quad F_1 = \frac{E_1}{\beta}.$$ 

D.8

The linearisation of the expression of $\frac{dF}{dt}$ is accomplished using logarithms:

$$\ln \left( -\frac{dF}{dt} \right) = \ln \left( \frac{F_1}{\tau_1} \right) - \frac{1}{\tau_1} t.$$
The plot of $\ln(-dF/dt)$ is shown in the graph below for a case where the derivative was obtained numerically (left) and using a graphic method (right).

For the left graph, the best straight line is $\ln(-dF/dt) = m_1 t + b_1$ where $m_1 = (-6.47 \pm 0.62) \times 10^{-4}$ and $b_1 = (-6.52 \pm 0.11)$, using $t$ in seconds and the force in gram-force units. If the derivative is computed numerically for unequal time intervals, the final parameters $E_1$ and $\tau_1$ are similar.

The best straight line for the right graph yields $m_1 = (-6.00 \pm 0.15) \times 10^{-4}$ and $b_1 = (-6.63 \pm 0.02)$ using $t$ in seconds and the force in gram-force units.

Thus, using the data from the left graph, $\tau_1 = \frac{1}{m_1} = 1546$ s and

$$F_1 = \tau_1 e^{b_1} = 2.28 \text{ gf} \Rightarrow E_1 = \beta F_1 = 7.39 \times 10^5 \text{ N m}^{-2}.$$ 

For the right graph, the final parameters are $\tau_1 = 1667$ s and $E_1 = 7.13 \times 10^5 \text{ N m}^{-2}$.

D.8

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<th>$\tau_1$ = 1546 s, $E_1$ = 7.39 × 10$^5$ N m$^{-2}$.</th>
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<td><img src="image" alt="Graphs" /></td>
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**Left:** $dF/dt$ computed numerically using equal time intervals. **Right:** using data from the graph in D.6.

D.9

For the 4 points on the left graph in D.8, we can write

$$F(t) = F_0 + F_1 e^{-t/\tau_1} \Rightarrow F_0 = F(t) - F_1 e^{-t/\tau_1}.$$ 

Thus, averaging $F_0$ for the 4 points of the left graph in D.8:

$$F_0 = \left( \frac{40.32 + 40.31 + 40.34 + 40.30}{4} \right) = 40.32 \text{ gf}$$

Finally,

$$E_0 = \beta F_0 = 324293 \times 40.32 \text{ N m}^{-2}.$$ 

D.9

$E_0 = 1.31 \times 10^7 \text{ N m}^{-2}$.
D.10

The function $y(t)$ is given by

$$y(t) = F(t) - F_0 - F_1 e^{-t/\tau_1},$$

and was added in the Table introduced in A.3 using $F_0 = 40.32$ gf, $F_1 = 2.28$ gf and $\tau_1 = 1546$ s.

D.10  See column $y(t)$ in the Table in A.3.

D.11

Since

$$y(t) = F(t) - F_0 - F_1 e^{-t/\tau_1},$$

then

$$y(t) = F_2 e^{-t/\tau_2} + F_3 e^{-t/\tau_3} + \cdots, \quad \tau_2 > \tau_3 > \cdots$$

At long times, when the contributions from the higher components are small enough, we expect a linear behaviour for $\ln y(t)$:

$$\ln y = \ln F_2 - \frac{1}{\tau_2} t.$$  

In this case, the $y(t)$ data points become meaningless above 500 s. In the region 200-500 s the graph is linear and that region can be used to extract the parameters of the second component. The equation of the straight line is $\ln y_2 = b_2 + m_2 t$. From the graph below,

$$m_2 = -(5.65 \pm 0.19) \times 10^{-3} \Rightarrow \tau_2 = \frac{1}{m_2} = 177 \text{s} \quad b_2 = 0.33 \pm 0.07 \Rightarrow F_2 = e^{b_2} = 1.39 \Rightarrow E_2 = \beta F_2 = 4.5 \times 10^5 \text{N m}^{-2}.$$ 

D.11

$$E_2 = 4.5 \times 10^5 \text{N m}^{-2}, \quad \tau_2 = 177 \text{s}.$$ 

The best straight line in the range 200-500 s yield the parameters $\tau_2$ and $E_2$ (Question D.11). The slope of the best straight line in the range [10, 30] s give an estimate of $\tau_3$ (Questions D.12 and D.13).
D.12

Below around 30 s there is clear deviation from a linear behaviour indicating the presence of a third component. In our case, the first data point was acquired at $t = 10$ s.

\[ t_i = 10 \text{ s} , \quad t_f = 30 \text{ s} \]

D.13

Drawing a line in the graph using the first data points (in the range defined in D.12), as shown in the graph in D.11, $\tau_3$ can be estimated as:

\[ m_3 = -0.02 \Rightarrow \tau_3 \approx m_3^{-1} , \]

\[ \tau_3 \approx 50 \text{ s} . \]

Part E. Measuring $E$ in constant stress conditions (0.6 points)

E.1

From Question C.1 we have

\[ \ell'_0 = (32.60 \pm 0.2) \text{ cm} . \]

The final length $\ell'$ should be measured. In our case,

\[ \ell' = 42.2 + 2 \times 0.5 = 43.2 \text{ cm} \Rightarrow \ell' = (43.2 \pm 0.2) \text{ cm} . \]

Therefore, the strain is

\[ \epsilon = \frac{\ell' - \ell'_0}{\ell'_0} = 0.325 . \]

Given that

\[ E = \frac{\sigma}{\epsilon} = \frac{Mg}{\pi R^2} = \frac{80.2 \times 10^{-3} \times 9.8}{\pi \times (0.24 \times 10^{-3})^2 \times 0.325} = 1.337 \times 10^7 \text{ N m}^{-2} . \]

Note that the radius $R$ of the stretched thread was not measured. We used the value measured in task B.4: $R \approx 0.24 \times 10^{-3}$ m.
The relative difference is

$$\frac{E - E_0}{E_0} = 0.021.$$  

\begin{equation}
E.1 \quad E = 1.337 \times 10^7 \text{ N m}^{-2}, \quad \frac{E - E_0}{E_0} = 2.1\%.
\end{equation}