In this problem we deal with a simplified model of accelerometers designed to activate the safety air bags of automobiles during a collision. We would like to build an electromechanical system in such a way that when the acceleration exceeds a certain limit, one of the electrical parameters of the system such as the voltage at a certain point of the circuit will exceed a threshold and the air bag will be activated as a result.

Note: Ignore gravity in this problem.

1 Consider a capacitor with parallel plates as in Figure 1. The area of each plate in the capacitor is A and the distance between the two plates is d. The distance between the two plates is much smaller than the dimensions of the plates. One of these plates is in contact with a wall through a spring with a spring constant k, and the other plate is fixed. When the distance between the plates is d the spring is neither compressed nor stretched, in other words no force is exerted on the spring in this state. Assume that the permittivity of the air between the plates is that of free vacuum ε_0. The capacitance corresponding to this distance between the plates of the capacitor is $C_0 = \varepsilon_0 A/d$. We put charges $+Q$ and $-Q$ on the plates and let the system achieve mechanical equilibrium.

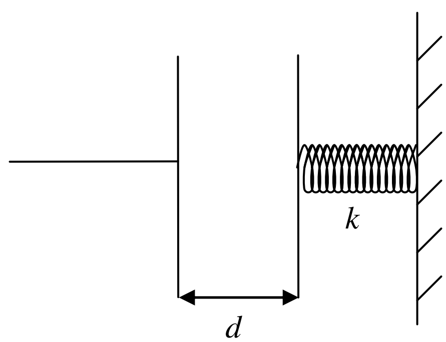


Figure 1

1.1	Calculate the electrical force, F_E, exerted by the plates on each other.	0.8
1.2	Let x be the displacement of the plate connected to the spring. Find x.	0.6
1.3	In this state, what is the electrical potential difference V between the plates of the capacitor in terms of Q, A, d, k?	0.4
1.4	Let C be the capacitance of the capacitor, defined as the ratio of charge to potential difference. Find C/C_0 as a function of Q, A, d and k.	0.3
1.5	What is the total energy, U, stored in the system in terms of Q, A, d and k?	0.6

Figure 2, shows a mass M which is attached to a conducting plate with negligible mass and also to two springs having identical spring constants k. The conducting plate can move back and forth in the space between two fixed conducting plates. All these plates are similar and have the same area A. Thus these three plates constitute two capacitors. As shown in Figure 2, the fixed plates are connected to the given potentials V and $-V$, and the middle plate is connected through a two-state switch to the ground. The wire connected to the movable plate does not disturb the motion of the plate and the three plates will always remain parallel. When the whole complex is not being accelerated, the distance from each fixed plate to the movable plate is d which is much smaller than the dimensions of the plates. The thickness of the movable plate can be ignored.

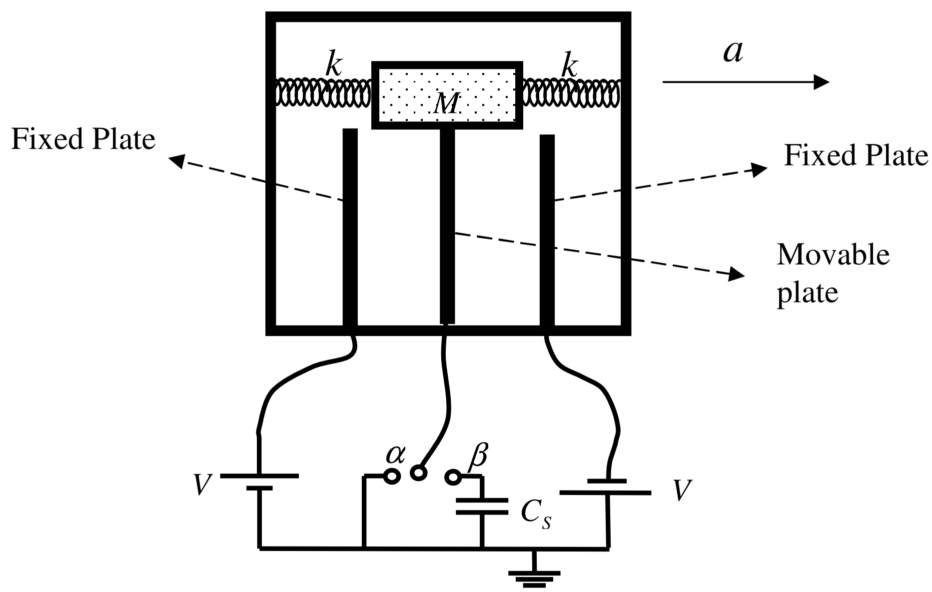


Figure 2

The switch can be in either one of the two states α and β. Assume that the capacitor complex is being accelerated along with the automobile, and the acceleration is constant. Assume that during this constant acceleration the spring does not oscillate and all components of this complex capacitor are in their equilibrium positions, i.e. they do not move with respect to each other, and hence with respect to the automobile.
Due to the acceleration, the movable plate will be displaced a certain amount x from the middle of the two fixed plates.

2 Consider the case where the switch is in state α i.e. the movable plate is connected to the ground through a wire, then

2.1	Find the charge on each capacitor as a function of x.	0.4
2.2	Find the net electrical force on the movable plate, F_E, as a function of x.	0.4
2.3	Assume $d >> x$ and terms of order x^2 can be ignored compared to terms of order d^2. Simplify the answer to the previous part.	0.2
2.4	Write the total force on the movable plate (the sum of the electrical and the spring forces) as $-k_{eff}x$ and give the form of k_{eff}.	0.7
2.5	Express the constant acceleration a as a function of x.	0.4

3 Now assume that the switch is in state β i.e. the movable plate is connected to the ground through a capacitor, the capacitance of which is C_S (there is no initial charge on the capacitors). If the movable plate is displaced by an amount x from its central position,

3.1	Find V_S the electrical potential difference across the capacitor C_S as a function of x.	1.5

3.2	Again assume that $d >> x$ and ignore terms of order x^2 compared to terms of order d^2. Simplify your answer to the previous part.	0.2

4 We would like to adjust the parameters in the problem such that the air bag will not be activated in normal braking but opens fast enough during a collision to prevent the driver's head from colliding with the windshield or the steering wheel. As you have seen in Part 2, the force exerted on the movable plate by the springs and the electrical charges can be represented as that of a spring with an effective spring constant k_{eff}. The whole capacitor complex is similar to a *mass and spring* system of mass M and spring constant k_{eff} under the influence of a constant acceleration a, which in this problem is the acceleration of the automobile.

Note: In this part of the problem, the assumption that the mass and spring are in equilibrium under a constant acceleration and hence are fixed relative to the automobile, no longer holds.

Ignore friction and consider the following numerical values for the parameters of the problem:

$$d = 1.0\,\text{cm},\quad A = 2.5\times10^{-2}\,\text{m}^2,\quad k = 4.2\times10^{3}\,\text{N/m},\quad \varepsilon_0 = 8.85\times10^{-12}\,\text{C}^2/\text{Nm}^2,$$
$$V = 12\,\text{V},\ M = 0.15\,\text{kg}.$$

4.1	Using this data, find the ratio of the electrical force you calculated in section 2.3 to the force of the springs and show that one can ignore the electrical forces compared to the spring forces.	0.6

Although we did not calculate the electrical forces for the case when the switch is in the state β, it can be shown that in this situation, quite similarly, the electrical forces are as small and can be ignored.

4.2	If the automobile while traveling with a constant velocity, suddenly brakes with a constant acceleration a, what is the maximum displacement of the movable plate? Give your answer in parameter.	0.6

Assume that the switch is in state β and the system has been designed such that when the electrical voltage across the capacitor reaches $V_S = 0.15\text{V}$, the air bag is activated. We would like the air bag not to be activated during normal braking when the automobile's acceleration is less than the acceleration of gravity $g = 9.8\,m/s^2$, but be activated otherwise.

4.3	How much should C_S be for this purpose?	0.6

We would like to find out if the air bag will be activated fast enough to prevent the driver's head from hitting the windshield or the steering wheel. Assume that as a result of collision, the automobile experiences a deceleration equal to g but the driver's head keeps moving at a constant speed.

4.4	By estimating the distance between the driver's head and the steering wheel, find the time t_1 it takes before the driver's head hits the steering wheel.	0.8

4.5	Find the time t_2 before the air bag is activated and compare it to t_1. Is the air bag activated in time? Assume that airbag opens instantaneously.	0.9

In physics, whenever we have an equality relation, both sides of the equation should be of the same type i.e. they must have the same dimensions. For example you cannot have a situation where the quantity on the right-hand side of the equation represents a length and the quantity on the left-hand side represents a time interval. Using this fact, sometimes one can nearly deduce the form of a physical relation without solving the problem analytically. For example if we were asked to find the time it takes for an object to fall from a height of h under the influence of a constant gravitational acceleration g, we could argue that one only needs to build a quantity representing a time interval, using the quantities g and h and the only possible way of doing this is $T = a(h/g)^{1/2}$. Notice that this solution includes an as yet undetermined coefficient a which is *dimensionless* and thus cannot be determined, using this method. This coefficient can be a number such as 1, $1/2$, $\sqrt{3}$, π, or any other real number. This method of deducing physical relations is called *dimensional analysis*. In dimensional analysis the dimensionless coefficients are not important and we do not need to write them. Fortunately in most physical problems these coefficients are of the order of 1 and eliminating them does not change the order of magnitude of the physical quantities. Therefore, by applying the dimensional analysis to the above problem, one obtains $T = (h/g)^{1/2}$.

Generally, the dimensions of a physical quantity are written in terms of the dimensions of four fundamental quantities: M (mass), L (length), T (time), and K (temperature). The dimensions of an arbitrary quantity, x is denoted by $[x]$. As an example, to express the dimensions of velocity v, kinetic energy E_k, and heat capacity C_V we write: $[v] = LT^{-1}$, $[E_k] = ML^2T^{-2}$, $[C_V] = ML^2T^{-2}K^{-1}$.

1 Fundamental Constants and Dimensional Analysis

1.1	Find the dimensions of *the fundamental constants,* i.e. the Planck's constant, h, the speed of light, c, the universal constant of gravitation, G, and the Boltzmann constant, k_B, in terms of the dimensions of length, mass, time, and temperature.	0.8

The Stefan-Boltzmann law states that the black body emissive power which is the total energy radiated per unit surface area of a black body in unit time is equal to $\sigma\theta^4$ where σ is the Stefan-Boltzmann's constant and θ is the absolute temperature of the black body.

1.2	Determine the dimensions of the Stefan-Boltzmann's constant in terms of the dimensions of length, mass, time, and temperature.	0.5

The Stefan-Boltzmann's constant is not a fundamental constant and one can write it in terms of fundamental constants i.e. one can write $\sigma = a\, h^{\alpha} c^{\beta} G^{\gamma} k_B^{\ \delta}$. In this relation a is a dimensionless parameter of the order of 1. As mentioned before, the exact value of a is not significant from our viewpoint, so we will set it equal to 1.

1.3	Find α, β, γ, and δ using dimensional analysis.	1.0

2 Physics of Black Holes

In this part of the problem, we would like to find out some properties of black holes using dimensional analysis. According to a certain theorem in physics known as the *no hair theorem*, all the characteristics of the black hole which we are considering in this problem depend only on the mass of the black hole. One characteristic of a black hole is the area of its *event horizon*. Roughly speaking, the event horizon is the boundary of the black hole. Inside this boundary, the gravity is so strong that even light cannot emerge from the region enclosed by the boundary.

We would like to find a relation between the mass of a black hole, m, and the area of its event horizon, A. This area depends on the mass of the black hole, the speed of light, and the universal constant of gravitation. As in 1.3 we shall write $A = G^{\alpha} c^{\beta} m^{\gamma}$.

2.1	Use dimensional analysis to find α, β, and γ.	0.8

From the result of 2.1 it becomes clear that the area of the event horizon of a black hole increases with its mass. From a classical point of view, nothing comes out of a black hole and therefore in all physical processes the area of the event horizon can only increase. In analogy with the second law of thermodynamics, Bekenstein proposed to assign entropy, S, to a black hole, proportional to the area of its event horizon i.e. $S = \eta A$. This conjecture has been made more plausible using other arguments.

2.2	Use the thermodynamic definition of entropy $dS = dQ/\theta$ to find the dimensions of entropy. dQ is the exchanged heat and θ is the absolute temperature of the system.	0.2

2.3	As in 1.3, express the dimensioned constant η as a function of the fundamental constants h, c, G, and k_B.	1.1

Do ***not*** *use dimensional analysis for the rest of problem, but you may use the results you have obtained in previous sections.*

3 Hawking Radiation

With a semi-quantum mechanical approach, Hawking argued that contrary to the classical point of view, black holes emit radiation similar to the radiation of a black body at a temperature which is called the *Hawking temperature*.

3.1	Use $E = mc^2$, which gives the energy of the black hole in terms of its mass, and the laws of thermodynamics to express the Hawking temperature θ_H of a black hole in terms of its mass and the fundamental constants. Assume that the black hole does no work on its surroundings.	0.8

3.2	The mass of an isolated black hole will thus change because of the Hawking radiation. Use Stefan-Boltzmann's law to find the dependence of this rate of change on the Hawking temperature of the black hole, θ_H and express it in terms of mass of the black hole and the fundamental constants.	0.7

3.3	Find the time t^*, that it takes an isolated black hole of mass m to evaporate completely i.e. to lose all its mass.	1.1

From the viewpoint of thermodynamics, black holes exhibit certain exotic behaviors. For example the heat capacity of a black hole is negative.

3.4	Find the heat capacity of a black hole of mass m.	0.6

4 Black Holes and the Cosmic Background Radiation

Consider a black hole exposed to the cosmic background radiation. The cosmic background radiation is a black body radiation with a temperature θ_B which fills the entire universe. An object with a total area A will thus receive an energy equal to $\sigma\theta_B^4 \times A$ per unit time. A black hole, therefore, loses energy through Hawking radiation and gains energy from the cosmic background radiation.

4.1	Find the rate of change of a black hole's mass, in terms of the mass of the black hole, the temperature of the cosmic background radiation, and the fundamental constants.	0.8
4.2	At a certain mass, m^*, this rate of change will vanish. Find m^* and express it in terms of θ_B and the fundamental constants.	0.4
4.3	Use your answer to 4.2 to substitute for θ_B in your answer to part 4.1 and express the rate of change of the mass of a black hole in terms of m, m^*, and the fundamental constants.	0.2
4.4	Find the Hawking temperature of a black hole at thermal equilibrium with cosmic background radiation.	0.4
4.5	Is the equilibrium stable or unstable? Why? (Express your answer mathematically)	0.6

Two stars rotating around their center of mass form a binary star system. Almost half of the stars in our galaxy are binary star systems. It is not easy to realize the binary nature of most of these star systems from Earth, since the distance between the two stars is much less than their distance from us and thus the stars cannot be resolved with telescopes. Therefore, we have to use either photometry or spectrometry to observe the variations in the intensity or the spectrum of a particular star to find out whether it is a binary system or not.

Photometry of Binary Stars

If we are exactly on the plane of motion of the two stars, then one star will occult (pass in front of) the other star at certain times and the intensity of the whole system will vary with time from our observation point. These binary systems are called ecliptic binaries.

1 Assume that two stars are moving on circular orbits around their common center of mass with a constant angular speed ω and we are exactly on the plane of motion of the binary system. Also assume that the surface temperatures of the stars are T_1 and T_2 $(T_1 > T_2)$, and the corresponding radii are R_1 and R_2 $(R_1 > R_2)$, respectively. The total intensity of light, measured on Earth, is plotted in Figure 1 as a function of time. Careful measurements indicate that the intensities of the incident light from the stars corresponding to the minima are respectively 90 and 63 percent of the maximum intensity, I_0, received from both stars $\left(I_0 = 4.8\times10^{-9}\ \text{W/m}^2\right)$. The vertical axis in Figure 1 shows the ratio I/I_0 and the horizontal axis is marked in days.

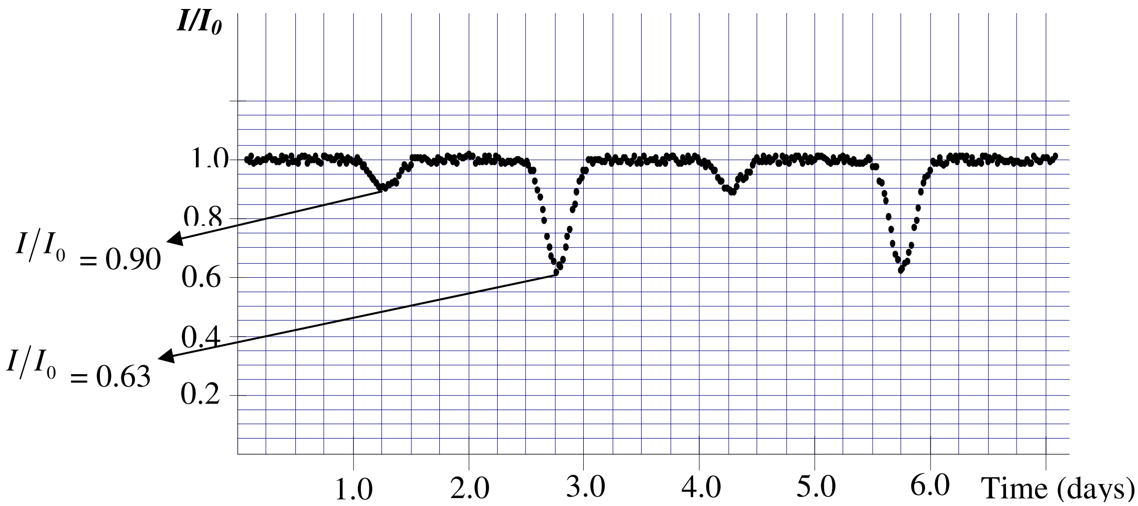


Figure 1. The relative intensity received from the binary star system as a function of time. The vertical axis has been scaled by $I_0 = 4.8\times10^{-9}\ \text{W/m}^2$. Time is given in days.

1.1	Find the period of the orbital motion. Give your answer in **seconds** up to two significant digits. What is the angular frequency of the system in **rad/sec**?	0.8

To a good approximation, the receiving radiation from a star is a uniform black body radiation from a flat disc with a radius equal to the radius of the star. Therefore, the power received from the star is proportional to AT^4 where A is area of the disc and T is the surface temperature of the star.

1.2	Use the diagram in Figure 1 to find the ratios T_1/T_2 and R_1/R_2.	1.6

Spectrometry of Binary Systems

In this section, we are going to calculate the astronomical properties of a binary star by using experimental spectrometric data of the binary system.

Atoms absorb or emit radiation at their certain characteristic wavelengths. Consequently, the observed spectrum of a star contains *absorption lines* due to the atoms in the star's atmosphere. Sodium has a characteristic yellow line spectrum (D_1 line) with a wavelength 5895.9Å (10 Å = 1 nm). We examine the absorption spectrum of atomic Sodium at this wavelength for the binary system of the previous section. The spectrum of the light that we receive from the binary star is Doppler-shifted, because the stars are moving with respect to us. Each star has a different speed. Accordingly the absorption wavelength for each star will be shifted by a different amount. Highly accurate wavelength measurements are required to observe the Doppler shift since the speed of the stars is much less than the speed of light. The speed of the center of mass of the binary system we consider in this problem is much smaller than the orbital velocities of the stars. Hence all the Doppler shifts can be attributed to the orbital velocity of the stars. Table 1 shows the measured spectrum of the stars in the binary system we have observed.

Table 1: Absorption spectrum of the binary star system for the Sodium D_1 line

t/days	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4
λ_1 (Å)	5897.5	5897.7	5897.2	5896.2	5895.1	5894.3	5894.1	5894.6
λ_2 (Å)	5893.1	5892.8	5893.7	5896.2	5897.3	5898.7	5899.0	5898.1

t/days	2.7	3.0	3.3	3.6	3.9	4.2	4.5	4.8
λ_1 (Å)	5895.6	5896.7	5897.3	5897.7	5897.2	5896.2	5895.0	5894.3
λ_2 (Å)	5896.4	5894.5	5893.1	5892.8	5893.7	5896.2	5897.4	5898.7

(*Note*: There is no need to make a graph of the data in this table)

2 Using Table 1,

2.1	Let v_1 and v_2 be the orbital velocity of each star. Find v_1 and v_2. The speed of light $c = 3.0\times10^8$ m/s. Ignore all relativistic effects.	1.8

2.2	Find the mass ratio of the stars (m_1/m_2).	0.7

2.3	Let r_1 and r_2 be the distances of each star from their center of mass. Find r_1 and r_2.	0.8

2.4	Let r be the distance between the stars. Find r.	0.2

3 The gravitational force is the only force acting between the stars.

3.1	Find the mass of each star up to one significant digit. The universal gravitational constant $G = 6.7\times10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$.	1.2

General Characteristics of Stars

4 Most of the stars generate energy through the same mechanism. Because of this, there is an empirical relation between their mass, M, and their luminosity, L, which is the total radiant power of the star. This relation could be written in the form $L/L_{Sun} = (M/M_{Sun})^{\alpha}$. Here, $M_{Sun} = 2.0\times10^{30}\ \mathrm{kg}$ is the solar mass and, $L_{Sun} = 3.9\times10^{26}\ \mathrm{W}$ is the solar luminosity. This relation is shown in a log-log diagram in Figure 2.

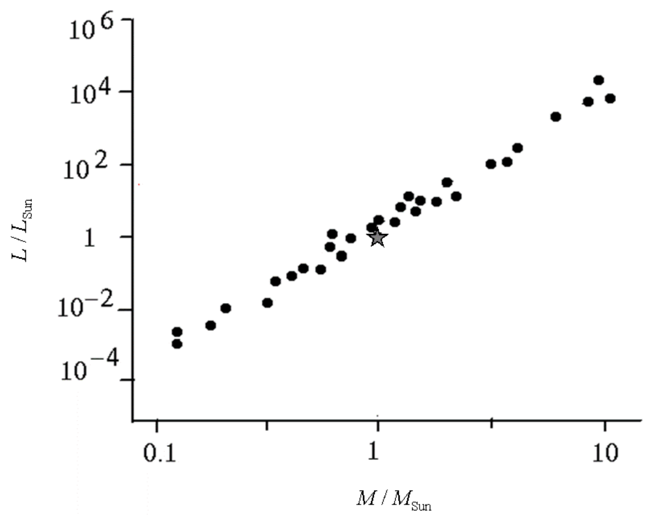


Figure 2. The luminosity of a star versus its mass varies as a power law. The diagram is log-log. The star-symbol represents Sun with a mass of $2.0\times10^{30}\ kg$ and luminosity of $3.9\times10^{26}\ \mathrm{W}$.

4.1	Find α up to one significant digit.	0.6

4.2	Let L_1 and L_2 be the luminosity of the stars in the binary system studied in the previous sections. Find L_1 and L_2.	0.6

4.3	What is the distance, d, of the star system from us in light years? To find the distance you can use the diagram of Figure 1. One light year is the distance light travels in one year.	0.9

4.4	What is the maximum angular distance, θ, between the stars from our observation point?	0.4

4.5	What is the smallest aperture size for an optical telescope, D, that can resolve these two stars?	0.4

Question "Orange"

1.1)
First of all, we use the Gauss's law for a single plate to obtain the electric field,

$E = \frac{\sigma}{\varepsilon_0}$. (0.2)

The density of surface charge for a plate with charge, Q and area, A is

$\sigma = \frac{Q}{A}$. (0.2)

Note that the electric field is resulted by two equivalent parallel plates. Hence the contribution of each plate to the electric field is $\frac{1}{2}E$. Force is defined by the electric filed times the charge, then we have

Force = $\frac{1}{2} E\, Q = \frac{Q^2}{2\varepsilon_0 A}$ (0.2)+ (0.2) (The ½ coefficient + the final result)

1.2)
The Hook's law for a spring is

$F_m = -k\,x$. (0.2)

In 1.2 we derived the electric force between two plates is

$F_e = \frac{Q^2}{2\varepsilon_0 A}$.

The system is stable. The equilibrium condition yields

$F_m = F_e$, (0.2)

$\Rightarrow \qquad x = \frac{Q^2}{2\varepsilon_0 A\,k}$ (0.2)

1.3)
The electric field is constant thus the potential difference, V is given by

$V = E(d - x)$ (0.2)

(Other reasonable approaches are acceptable. For example one may use the definition of capacity to obtain V.)
By substituting the electric field obtained from previous section to the above equation, we get, $V = \frac{Q\,d}{\varepsilon_0 A}\left(1 - \frac{Q^2}{2\varepsilon_0 A\,k\,d}\right)$ (0.2)

1.4)
C is defined by the ratio of charge to potential difference, then

$C = \frac{Q}{V}$. (0.1)

Using the answer to 1.3, we get $\dfrac{C}{C_0} = \left(1 - \dfrac{Q^2}{2\varepsilon_0 A k d}\right)^{-1}$ (0.2)

1.5)

Note that we have both the mechanical energy due to the spring

$$U_m = \frac{1}{2} k x^2, \qquad (0.2)$$

and the electrical energy stored in the capacitor.

$$U_E = \frac{Q^2}{2C}\ . \qquad (0.2)$$

Therefore the total energy stored in the system is

$$U = \frac{Q^2 d}{2\varepsilon_0 A}\left(1 - \frac{Q^2}{4\varepsilon_0 A k d}\right) \qquad (0.2)$$

2.1)

For the given value of x, the amount of charge on each capacitor is

$$Q_1 = V\,C_1 = \frac{\varepsilon_0 A V}{d - x}\ , \qquad (0.2)$$

$$Q_2 = V\,C_2 = \frac{\varepsilon_0 A V}{d + x}\ . \qquad (0.2)$$

2.2)

Note that we have two capacitors. By using the answer to 1.1 for each capacitor, we get

$$F_1 = \frac{Q_1^{\ 2}}{2\varepsilon_0 A},$$

$$F_2 = \frac{Q_2^{\ 2}}{2\varepsilon_0 A}.$$

As these two forces are in the opposite directions, the net electric force is

$$F_E = F_1 - F_2, \quad (0.2) \qquad \Rightarrow \qquad F_E = \frac{\varepsilon_0 A V^2}{2}\left(\frac{1}{(d-x)^2} - \frac{1}{(d+x)^2}\right) \quad (0.2)$$

2.3)

Ignoring terms of order x^2 in the answer to 2.2., we get

$$F_E = \frac{2\varepsilon_0 A V^2}{d^3} x \qquad (0.2)$$

2.4)

There are two springs placed in series with the same spring constant, k, then the mechanical force is

$F_m = -2kx$.   (The coefficient (2) has (0.2))

Combining this result with the answer to 2.4 and noticing that these two forces are in the opposite directions, we get

$F = F_m + F_E$, $\Rightarrow$ $F = -2\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)x$, (Opposite signs of the two forces have (0.3))

$\Rightarrow \quad k_{eff} = 2\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)$ (0.2)

2.5)

By using the Newtown's second law,

$F = ma$ (0.2)

and the answer to 2.4, we get

$a = -\frac{2}{m}\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)x$ (0.2)

3.1)

Starting with Kirchhoff's laws, for two electrical circuits, we have

$$\begin{cases} \frac{Q_S}{C_S} + V - \frac{Q_2}{C_2} = 0 \\ -\frac{Q_S}{C_S} + V - \frac{Q_1}{C_1} = 0 \\ Q_2 - Q_1 + Q_S = 0 \end{cases}$$

(Each has (0.3), Note: the sings may depend on the specific choice made)

Noting that $V_s = \frac{Q_S}{C_S}$ one obtains

$$\Rightarrow V_S = V \frac{\frac{2\varepsilon_0 A x}{d^2 - x^2}}{C_S + \frac{2\varepsilon_0 A d}{d^2 - x^2}}.$$

((0.4) + (0.2): (0.4) for solving the above equations and (0.2) for final result)

Note: Students may simplify the above relation using the approximation $d^2 >> x^2$. It does not matter in this section.

3.2)

Ignoring terms of order x^2 in the answer to 3.1., we get

$$V_S = V \frac{2\varepsilon_0 A x}{d^2 C_S + 2\varepsilon_0 A d} \; . \qquad (0.2)$$

4.1)

The ratio of the electrical force to the mechanical (spring) force is

$$\frac{F_E}{F_m} = \frac{\varepsilon_0 A V^2}{k d^3} \; ,$$

Putting the numerical values:

$$\frac{F_E}{F_m} = 7.6 \times 10^{-9} \; .$$

((0.2) + (0.2) + (0.2): (0.2) for order of magnitude, (0.2) for two significant digits and (0.2) for correct answer (7.6 or 7.5)).

As it is clear from this result, we can ignore the electrical forces compared to the electric force.

4.2)

As seen in the previous section, one may assume that the only force acting on the moving plate is due to springs:

$F = 2k x$. (The concept of equilibrium (0.2))

Hence in mechanical equilibrium, the displacement of the moving plate is

$$x = \frac{ma}{2k} \; .$$

The maximum displacement is twice this amount, like the mass spring system in a gravitational force field, when the mass is let to fall.

$x_{\max} = 2x$ (0.2)

$$x_{\max} = \frac{m a}{k} \qquad (0.2)$$

4.3)

At the acceleration

$a = g$, (0.2)

The maximum displacement is

$$x_{\max} = \frac{m g}{k} \; .$$

Moreover, from the result obtained in 3.2, we have

$$V_S = V\frac{2\varepsilon_0 A\, x_{\max}}{d^2 C_S + 2\varepsilon_0 A\, d}$$

This should be the same value given in the problem, $0.15\,\mathrm{V}$.

$$\Rightarrow \qquad C_S = \frac{2\varepsilon_0 A}{d}\left(\frac{V\, x_{\max}}{V_S\, d} - 1\right) \qquad (0.2)$$

$$\Rightarrow \qquad C_S = 8.0\times 10^{-11}\,\mathrm{F} \qquad (0.2)$$

4.4)

Let ℓ be the distance between the driver's head and the steering wheel. It can be estimated to be about

$\ell = 0.4\,m - 1\,m$. (0.2)

Just at the time the acceleration begins, the relative velocity of the driver's head with respect to the automobile is zero.

$\Delta v(t=0) = 0$, (0.2)

then

$$\ell = \frac{1}{2} g\, t_1^{\,2} \qquad \Rightarrow \qquad t_1 = \sqrt{\frac{2\,\ell}{g}} \qquad (0.2)$$

$t_1 = 0.3 - 0.5\ s$ (0.2)

4.5)

The time t_2 is half of period of the harmonic oscillator, hence

$$t_2 = \frac{T}{2}, \qquad (0.3)$$

The period of harmonic oscillator is simply given by

$$T = 2\pi\sqrt{\frac{m}{2k}}\ , \qquad (0.2)$$

therefore,

$t_2 = 0.013\ s$. (0.2)

As $t_1 > t_2$, the airbag activates in time. (0.2)

1.1) One may use any reasonable equation to obtain the dimension of the questioned quantities.

I) The Planck relation is $h\nu = E \quad \Rightarrow \quad [h][\nu] = [E] \quad \Rightarrow \quad [h] = [E][\nu]^{-1} = ML^2T^{-1}$
(0.2)

II) $[c] = LT^{-1}$ (0.2)

III) $F = \dfrac{Gmm}{r^2} \quad \Rightarrow \quad [G] = [F][r^2][m]^{-2} = M^{-1}L^3T^{-2}$ (0.2)

IV) $E = K_B\theta \quad \Rightarrow \quad [K_B] = [\theta]^{-1}[E] = ML^2T^{-2}K^{-1}$ (0.2)

1.2) Using the Stefan-Boltzmann's law,

$\dfrac{Power}{Area} = \sigma\theta^4$, or any equivalent relation, one obtains:

(0.3)

$[\sigma]K^4 = [E]L^{-2}T^{-1} \Rightarrow [\sigma] = MT^{-3}K^{-4}.$ (0.2)

1.3) The Stefan-Boltzmann's constant, up to a numerical coefficient, equals $\sigma = h^\alpha c^\beta G^\gamma k_B{}^\delta$, where $\alpha, \beta, \gamma, \delta$ can be determined by dimensional analysis. Indeed, $[\sigma] = [h]^\alpha [c]^\beta [G]^\gamma [k_B]^\delta$, where e.g. $[\sigma] = MT^{-3}K^{-4}$.

$$MT^{-3}K^{-4} = \left(ML^2T^{-1}\right)^\alpha \left(LT^{-1}\right)^\beta \left(M^{-1}L^3T^{-2}\right)^\gamma \left(ML^2T^{-2}K^{-1}\right)^\delta = M^{\alpha-\gamma+\delta}L^{2\alpha+\beta+3\gamma+2\delta}T^{-\alpha-\beta-2\gamma-2\delta}K^{-\delta},$$

(0.2)

The above equality is satisfied if,

$$\Rightarrow \begin{cases} \alpha - \gamma + \delta = 1, \\ 2\alpha + \beta + 3\gamma + 2\delta = 0, \\ -\alpha - \beta - 2\gamma - 2\delta = -3, \\ -\delta = -4, \end{cases}$$ (Each one (0.1)) $$\Rightarrow \begin{cases} \alpha = -3, \\ \beta = -2, \\ \gamma = 0, \\ \delta = 4. \end{cases}$$ (Each one (0.1))

$$\Rightarrow \qquad \sigma = \frac{k_B{}^4}{c^2h^3}.$$

2.1) Since A, the area of the event horizon, is to be calculated in terms of m from a classical theory of relativistic gravity, e.g. the General Relativity, it is a combination of c, characteristic of special relativity, and G characteristic of gravity. Especially, it is independent of the Planck constant h which is characteristic of quantum mechanical phenomena.

$$A = G^{\alpha} c^{\beta} m^{\gamma}$$

Exploiting dimensional analysis,

$$\Rightarrow \ [A] = [G]^{\alpha}[c]^{\beta}[m]^{\gamma} \ \Rightarrow \ L^2 = \left(M^{-1}L^3T^{-2}\right)^{\alpha}\left(LT^{-1}\right)^{\beta} M^{\gamma} = M^{-\alpha+\gamma}L^{3\alpha+\beta}T^{-2\alpha-\beta}$$

(0.2)

The above equality is satisfied if,

$$\Rightarrow \begin{cases} -\alpha+\gamma = 0, \\ 3\alpha+\beta = 2, \\ -2\alpha-\beta = 0, \end{cases} \text{(Each one (0.1))} \quad \Rightarrow \quad \begin{cases} \alpha = 2, \\ \beta = -4, \\ \gamma = 2, \end{cases} \text{(Each one (0.1))} \Rightarrow$$

$$A = \frac{m^2 G^2}{c^4}.$$

2.2)

From the definition of entropy $dS = \dfrac{dQ}{\theta}$, one obtains $[S] = [E][\theta]^{-1} = ML^2T^{-2}K^{-1}$ (0.2)

2.3) Noting $\eta = S/A$, one verifies that,

$$\begin{cases} [\eta] = [S][A]^{-1} = MT^{-2}K^{-1}, \\ [\eta] = [G]^{\alpha}[h]^{\beta}[c]^{\gamma}[k_B]^{\delta} = M^{-\alpha+\beta+\delta}L^{3\alpha+2\beta+\gamma+2\delta}T^{-2\alpha-\beta-\gamma-2\delta}K^{-\delta}, \end{cases}$$

(0.2)

Using the same scheme as above,

$$\Rightarrow \begin{cases} -\alpha+\beta+\delta = 1, \\ 3\alpha+2\beta+\gamma+2\delta = 0, \\ -2\alpha-\beta-\gamma-2\delta = -2, \\ \delta = 1, \end{cases} \text{(Each one (0.1))} \quad \Rightarrow \begin{cases} \alpha = -1, \\ \beta = -1, \\ \gamma = 3, \\ \delta = 1, \end{cases} \text{(Each one (0.1))}$$

thus, $\eta = \dfrac{c^3 k_B}{G h}$. (0.1)

3.1)

The first law of thermodynamics is $dE = dQ + dW$. By assumption, $dW = 0$. Using the definition of entropy, $dS = \frac{dQ}{\theta}$, one obtains,

$$dE = \theta_H \, dS + 0,$$ (0.2) + (0.1), for setting $dW = 0$.

Using, $\begin{cases} S = \frac{Gk_B}{ch} m^2, \\ E = mc^2, \end{cases}$ [(0.1) for *S*]

one obtains, $\theta_H = \frac{dE}{dS} = \left(\frac{dS}{dE}\right)^{-1} = c^2 \left(\frac{dS}{dm}\right)^{-1}$ (0.2)

Therefore, $\theta_H = \left(\frac{1}{2}\right) \frac{c^3 h}{Gk_B} \frac{1}{m}$. (0.1)+(0.1) (for the coefficient)

3.2) The Stefan-Boltzmann's law gives the rate of energy radiation per unit area. Noting that $E = mc^2$ we have:

$$\begin{cases} dE/dt = -\sigma \theta_H{}^4 A, & \text{(0.2)} \\ \sigma = \frac{k_B{}^4}{c^2 h^3}, \\ A = \frac{m^2 G^2}{c^4} \\ E = mc^{2,} \end{cases} \Rightarrow c^2 \frac{dm}{dt} = -\frac{k_B{}^4}{c^2 h^3} \left(\frac{c^3 h}{2G k_B} \frac{1}{m}\right)^4 \frac{m^2 G^2}{c^4},$$ (0.2)

$$\Rightarrow \quad \frac{dm}{dt} = -\frac{1}{16} \frac{c^4 h}{G^2} \frac{1}{m^2}.$$ (0.1) (for simplification) + (0.2) (for the minus sign)

3.3)

By integration:

$$\frac{dm}{dt} = -\frac{1}{16} \frac{c^4 h}{G^2} \frac{1}{m^2}. \quad \Rightarrow \int m^2 dm = -\int \frac{c^4 h}{16G^2} dt$$ (0.3)

$$\Rightarrow m^3(t) - m^3(0) = -\frac{3c^4 h}{16G^2} t,$$ (0.2) + (0.2) (Integration and correct boundary values)

At $t = t^*$ the black hole evaporates completely:

$m(t^*) = 0$ (0.1) $\Rightarrow t^* = \frac{16G^2}{3c^4 h} m^3$ (0.2)+(0.1) (for the coefficient)

3.4) C_V measures the change in E with respect to variation of θ.

$$\begin{cases} C_V = \dfrac{dE}{d\theta}, & (0.2) \\ E = mc^2, & (0.2) \\ \theta = \dfrac{c^3 h}{2Gk_B}\dfrac{1}{m} \end{cases} \Rightarrow \quad C_V = -\frac{2Gk_B}{ch} m^2.$$

0.1)+(0.1) (for the coefficient)

4.1) Again the Stefan-Boltzmann's law gives the rate of energy loss per unit area of the black hole. A similar relation can be used to obtain the energy gained by the black hole due to the background radiation. To justify it, note that in the thermal equilibrium, the total change in the energy is vanishing. The blackbody radiation is given by the Stefan-Boltzmann's law. Therefore the rate of energy gain is given by the same formula.

(0.1) + (0.4) (For the first and the second terms respectively)

$$\begin{cases} \dfrac{dE}{dt} = -\sigma\theta^4 A + \sigma\theta_B{}^4 A \\ E = mc^2, \end{cases} \Rightarrow \quad \frac{dm}{dt} = -\frac{hc^4}{16G^2}\frac{1}{m^2} + \frac{G^2}{c^8 h^3}(k_B\,\theta_B)^4 m^2 \qquad (0.3)$$

4.2)

Setting $\dfrac{dm}{dt} = 0$, we have:

$$-\frac{hc^4}{16G^2}\frac{1}{m^{*2}} + \frac{G^2}{c^8 h^3}(k_B\,\theta_B)^4 m^{*2} = 0 \qquad (0.2)$$

and consequently,

$$m^* = \frac{c^3 h}{2G\,k_B}\frac{1}{\theta_B} \qquad (0.2)$$

4.3)

$$\theta_B = \frac{c^3 h}{2G\,k_B}\frac{1}{m^*} \quad \Rightarrow \quad \frac{dm}{dt} = -\frac{hc^4}{16G^2}\frac{1}{m^2}\left(1 - \frac{m^4}{m^{*4}}\right) \qquad (0.2)$$

4.4) Use the solution to 4.2,

$m^* = \dfrac{c^3 h}{2G\,k_B}\dfrac{1}{\theta_B}$ (0.2) and 3.1 to obtain, $\theta^* = \dfrac{c^3 h}{2Gk_B}\dfrac{1}{m^*} = \theta_B$ (0.2)

One may also argue that m^* corresponds to thermal equilibrium. Thus for $m = m^*$ the black hole temperature equals θ_B.

Or one may set $\dfrac{dE}{dt} = -\sigma\left(\theta^{*4} - \theta_B{}^4\right)A = 0$ to get $\theta^* = \theta_B$.

4.5) Considering the solution to 4.3, one verifies that it will go away from the equilibrium. (0.6)

$$\frac{dm}{dt} = -\frac{hc^4}{G^2}\frac{1}{m^2}\left(1 - \frac{m^4}{m^{*4}}\right) \Rightarrow \begin{cases} m > m^* & \Rightarrow \quad \dfrac{dm}{dt} > 0 \\ m < m^* & \Rightarrow \quad \dfrac{dm}{dt} < 0 \end{cases}$$

Question "Pink"

1.1

Period = 3.0 days = $2.6\times10^5\,\text{s}$. (0.4)

Period = $\dfrac{2\pi}{\omega}$ (0.2) $\Rightarrow$ $\omega = 2.4\times10^{-5}$ rad s^{-1}. (0.2)

1.2

Calling the minima in the diagram 1, $I_1/I_0 = \alpha = 0.90$ and $I_2/I_0 = \beta = 0.63$, we have:

$$\frac{I_0}{I_1} = 1 + \left(\frac{R_2}{R_1}\right)^2\left(\frac{T_2}{T_1}\right)^4 = \frac{1}{\alpha} \quad (0.4)$$

$$\frac{I_2}{I_1} = 1 - \left(\frac{R_2}{R_1}\right)^2\left(1-\left(\frac{T_2}{T_1}\right)^4\right) = \frac{\beta}{\alpha} \quad (0.4) \quad \text{(or equivalent relations)}$$

From above, one finds:

$\dfrac{R_1}{R_2} = \sqrt{\dfrac{\alpha}{1-\beta}} \Rightarrow \dfrac{R_1}{R_2} = 1.6$ (0.2+0.2) and $\dfrac{T_1}{T_2} = \sqrt[4]{\dfrac{1-\beta}{1-\alpha}} \Rightarrow \dfrac{T_1}{T_2} = 1.4$ (0.2+0.2)

2.1)

Doppler-Shift formula:

$\dfrac{\Delta\lambda}{\lambda_0} \cong \dfrac{v}{c}$ (or equivalent relation) (0.4)

Maximum and minimum wavelengths: $\lambda_{1,\max} = 5897.7$ Å, $\lambda_{1,\min} = 5894.1$ Å

$\lambda_{2,\max} = 5899.0$ Å, $\lambda_{2,\min} = 5892.8$ Å

Difference between maximum and minimum wavelengths:

$\Delta\lambda_1 = 3.6$ Å , $\Delta\lambda_2 = 6.2$ Å (All 0.6)

Using the Doppler relation and noting that the shift is due to twice the orbital speed: (Factor of two 0.4)

$$v_1 = c\frac{\Delta\lambda_1}{2\lambda_0} = 9.2\times10^4 \text{ m/s} \quad (0.2)$$

$$v_2 = c\frac{\Delta\lambda_2}{2\lambda_0} = 1.6\times10^5 \text{ m/s} \quad (0.2)$$

The student can use the wavelength of central line and maximum (or minimum) wavelengths. Marking scheme is given in the Excel file.

2.2) As the center of mass is not moving with respect to us: (0.5)

$$\frac{m_1}{m_2} = \frac{v_2}{v_1} = 1.7 \quad (0.2)$$

2.3)

Writing $r_i = \dfrac{v_i}{\omega}$ for $i = 1,2$, we have (0.4)

$r_1 = 3.8\times10^9$ m, (0.2) $\qquad r_2 = 6.5\times10^9$ m (0.2)

2.4)

$r = r_1 + r_2 = 1.0\times10^{10}$ m (0.2)

3.1)

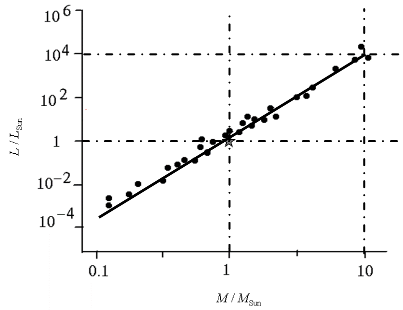
The gravitational force is equal to mass times the centrifugal acceleration

$$G\frac{m_1 m_2}{r^2} = m_1\frac{v_1^{\,2}}{r_1} = m_2\frac{v_2^{\,2}}{r_2} \quad (0.7)$$

Therefore,

$$\begin{cases} m_1 = \dfrac{r^2 v_2^{\,2}}{G r_2} \\ m_2 = \dfrac{r^2 v_1^{\,2}}{G r_1} \end{cases} \quad (0.1) \quad \Rightarrow \quad \begin{cases} m_1 = 6\times10^{30}\,\text{kg} \\ m_2 = 3\times10^{30}\,\text{kg} \end{cases} \quad (0.2 + 0.2)$$

4.1) As it is clear from the diagram, with one significant digit, $\alpha = 4$. (0.6)



4.2)

As we have found in the previous section: $L_i = L_{Sun}\left(\frac{M_i}{M_{Sun}}\right)^4$ (0.2)

So,

$$L_1 = 3\times10^{28}\,\text{Watt} \ (0.2)$$
$$L_2 = 4\times10^{27}\,\text{Watt} \ (0.2)$$

4.3) The total power of the system is distributed on a sphere with radius d to produce I_0, that is:

$$I_0 = \frac{L_1 + L_2}{4\pi d^2} \quad (0.5) \qquad \Rightarrow d = \sqrt{\frac{L_1 + L_2}{4\pi I_0}} = 1\times10^{18}\,\text{m} \quad (0.2)$$

= 100 ly. (0.2)

4.4) $$\theta \cong \tan\theta = \frac{r}{d} = 1\times10^{-8}\,\text{rad.} \quad (0.2 + 0.2)$$

4.5)

A typical optical wavelength is λ_0. Using uncertainty relation:

$$D = \frac{d\,\lambda_0}{r} \cong 50\text{ m.} \quad (0.2 + 0.2)$$